

Topological triviality of flat Hamiltonians

Pratik Sathe^{1,2,3,*} and Rahul Roy^{2,†}

¹Theoretical Division, T-4, *Los Alamos National Laboratory*, Los Alamos, New Mexico 87545, USA

²Mani L. Bhaumik Institute for Theoretical Physics, Department of Physics and Astronomy,

University of California Los Angeles, Los Angeles, California 90095, USA

³Information Science and Technology Institute, *Los Alamos National Laboratory*, Los Alamos, New Mexico 87545, USA



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Landau levels play a key role in theoretical models of the quantum Hall effect. Each Landau level is degenerate, flat, and topologically nontrivial. Motivated by Landau levels, we study tight-binding Hamiltonians whose energy levels are all flat. We demonstrate that in two dimensions, for such Hamiltonians, the flat bands must be topologically trivial. To that end, we show that the projector onto each flat band is necessarily strictly local. Our conclusions do not need the assumption of lattice translational invariance.

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Introduction. Topological phases of matter are a cornerstone of modern condensed matter physics. Beginning with the discovery of the quantum Hall effects (QHE) [1,2], topology has played an important role in understanding and predicting novel phases of matter ranging from topological insulators and superconductors to topologically ordered phases of matter [3,4].

The Hall conductivity of a filled Landau level or a Bloch band is proportional to a topological invariant called the Chern number [5–7]. Much like Landau levels, a Bloch band that is flat or dispersionless and which has a nonzero Chern number can host fractional quantum Hall phases when subject to interactions. Such systems are known as fractional Chern insulators (FCIs) and hold significant experimental appeal since they have the potential to exhibit fractional QHE at small (or even zero) magnetic fields and high temperatures [8]. Consequently, a substantial amount of research has focused on the study of topological flat-band models (see Refs. [9–11] for reviews).

Hamiltonians with flat bands are also interesting because of numerous other phenomena they exhibit [12]. The effects of interactions are pronounced in flat-band Hamiltonians, and hence they serve as a platform to explore novel correlated phases of matter. In addition to FCIs, some prominent examples include unconventional superconductivity in bilayer graphene [13–16], flat-band superfluidity [17,18], and flat-band ferromagnetism [19].

Various no-go theorems restrict the possibility of topological flat bands. Strict localization of a band projector, a condition more restrictive than band flatness, was shown to imply a vanishing Hall conductance [20]. Further, if a strictly local (SL) Hamiltonian has a flat band, then it must have a Chern number of zero [21,22] (see also Ref. [23]). These results apply to Hamiltonians with lattice translational

invariance (LTI). It has also been shown that local commuting projector Hamiltonians cannot exhibit the QHE [24,25]. The noninteracting limit of such a model has a flat band that is spanned by an orthogonal basis of compactly supported wave functions [26,27].

Noninteracting Hamiltonians that exclusively have flat bands, which we call flat Hamiltonians or all-bands-flat Hamiltonians, have also attracted significant attention. Flat Hamiltonians can exhibit unconventional behavior when interactions are introduced, such as many-body localization without LTI [28,29], nonlinear caging [30,31], and transport via two-particle bound states [32,33]. Furthermore, the spectrum of a flat Hamiltonian is similar to Landau levels since all the energies are flat and degenerate. Consequently, topological flat Hamiltonians that have SL hoppings seem to be particularly promising as platforms for fractional QHE. Indeed, many of the initial proposals for FCIs were Hamiltonians with all bands being approximately flat [34–37].

Naively, no-go theorems from Refs. [21,22] would suggest that flat Hamiltonians should be topologically trivial. However, flat Hamiltonians *can* be topological [38], in apparent contradiction to these theorems. We identify the exotic conditions required for such models to be topological. To that end, we prove some unique properties of SL flat Hamiltonians and establish a no-go theorem. First, we show that the projection operators associated with all the flat bands in a SL flat Hamiltonian are also SL, regardless of lattice dimension or LTI. Next, we show that the spectrum of such a Hamiltonian is unchanged with twisted boundary conditions. Using properties unique to SL projectors, we show that each flat band in a flat Hamiltonian has a Chern number of zero, regardless of LTI. The no-go theorem is stated below.

Theorem. Consider a SL tight-binding Hamiltonian defined on a two-dimensional (2D) system of size $L_x \times L_y$ cells with periodic boundary conditions. Let n denote the number of distinct energies and R denote the maximum hopping range of the Hamiltonian. If $L_{x,y} \geq 3nR$, then the Chern number associated with each energy is 0.

*Contact author: psathe@lanl.gov

†Contact author: rroy@physics.ucla.edu

Within the context of flat Hamiltonians with LTI, the inequality above is broadly satisfied. However, topological flat Hamiltonians must break the condition above (as exemplified by the models in Ref. [38]).

While our focus is on flat Hamiltonians, we emphasize that our conclusions go beyond flat Hamiltonians. Specifically, we also show that if a band or energy has a SL projector, then its Chern number is zero, independent of the flatness of the remaining bands.

Notation and Setup. We consider 2D tight-binding Hamiltonians with orbitals $|\mathbf{r}, \alpha\rangle$, where $\mathbf{r} = (x, y)$ denotes cell position and $\alpha = 1, \dots, N$ denotes different orbitals in a unit cell. We will restrict our discussion to systems that lie on a torus (i.e., those that satisfy periodic boundary conditions). Without loss of generality, we consider square lattices of size $L_x \times L_y$ with lattice constant 1.

The concepts of strict localization will play an important role in our arguments. A wave function is compactly supported if it has nonzero support only on a finite region of the lattice. An operator O in an infinite system is said to be strictly local (SL) if $\langle \mathbf{r}, \alpha | O | \mathbf{r}', \beta \rangle = 0$ (for any α, β) whenever $|\mathbf{r} - \mathbf{r}'| > R$ for some finite number R . We call the smallest value of R for which this is true the maximum hopping range of the operator. For finite-sized systems, this condition is satisfied by every operator. Therefore, we propose the following definition for finite systems on a torus: an operator is called SL if it does not connect any two cells that are maximally separated from each other. This is equivalent to the conditions $2R + 2 \leq L_{x/y}$.

Let us now define flat Hamiltonians. We recall that in an infinite-sized system, the spectrum of a Hamiltonian with LTI consists of a finite number of continuous bands following Bloch's theorem. The energy eigenvalues of a Hamiltonian on a finite system are necessarily discrete but are still arranged in the form of bands if the Hamiltonian has LTI. If a Hamiltonian with LTI has only flat bands, then we call it a flat Hamiltonian. Thus, each band has a degeneracy equal to the system size. In the case of Hamiltonians without LTI, we consider those that have a similar property of possessing highly degenerate energy levels. In a slight abuse of notation, we will refer to the distinct energy levels of such Hamiltonians as flat bands even in the absence of LTI. Regardless of LTI, all our results are derived for Hamiltonians and system sizes that satisfy $3nR \leq L_{x,y}$, where n denotes the number of distinct energies or flat bands, and R denotes the maximum hopping range of H [39].

Each energy eigenspace of a Hamiltonian may be associated with an orthogonal projector, i.e., a Hermitian operator P that satisfies $P^2 = P$. The locality properties of projectors associated with energy eigenspaces will form an important part of our subsequent discussion.

We will also find it useful to use a simple kind of dimensional reduction. Specifically, any 2D SL operator can always be regarded as a 1D SL operator by grouping together all the lattice sites in every row (or column) into a single 1D lattice site. For example, a 2D next-nearest-neighbor (NNN) hopping projector P (i.e., one with $R = \sqrt{2}$) is also a nearest-neighbor (NN) hopping projector (with $R = 1$) on a 1D lattice with one truncated dimension.

We will find it useful to define ‘‘hopping matrices’’ for 1D SL projectors. Specifically, we define P_i^x for a SL projector P

as follows:

$$P_i^x := \langle x + i | P | x \rangle. \quad (1)$$

The number of rows or columns in this matrix is equal to the number of orbitals in each cell. Such hopping matrices can also be defined for 2D SL projectors using the dimensional reduction described above.

Flat Hamiltonians have strictly local projectors. We will now show that if a SL Hamiltonian is flat, then all the projectors are also SL. Furthermore, if the maximum hopping range of the Hamiltonian is R and the number of distinct energies is n , then the maximum hopping range of each projector is less than or equal to nR .

To that end, consider a Hamiltonian H with energy eigenvalues $E_1 < E_2 < \dots < E_n$. Let the corresponding projectors be denoted by $P_1, P_2, \dots, P_n$. Without loss of generality, all the energies can be chosen to be nonzero. Taking the m th power of H , we get

$$H^m = \sum_{i=1}^n E_i^m P_i \quad (2)$$

for all integers $m \geq 1$. This follows from the fact that $P_i P_j = P_j P_i = P_i \delta_{i,j}$ for all i, j .

For any m , H^m has a maximum hopping range of no more than mR . Therefore, each of $H, \dots, H^n$ has vanishing matrix elements between any two locations separated by a distance greater than nR . For concreteness, consider orbitals α and β at any two cells $\vec{r}_1$ and $\vec{r}_2$, respectively, such that $|\vec{r}_1 - \vec{r}_2| > nR$. Let $j \equiv (\vec{r}_1, \alpha)$ and $k \equiv (\vec{r}_2, \beta)$ label the orbitals. Equating the (j, k) matrix elements of both sides of Eqs. (2) for $m = 1, \dots, n$ gives us

$$\underbrace{\begin{pmatrix} E_1 & E_2 & \dots & E_n \\ E_1^2 & E_2^2 & \dots & E_n^2 \\ \vdots & & & \vdots \\ E_1^n & E_2^n & \dots & E_n^n \end{pmatrix}}_M \begin{pmatrix} (P_1)_{j,k} \\ (P_2)_{j,k} \\ \vdots \\ (P_n)_{j,k} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (3)$$

The only solution of (3) is the zero vector since

$$\det M = \prod_{i=1}^n E_i \prod_{1 \leq j < k \leq n} (E_j - E_k) \neq 0,$$

where the second product series is the Vandermonde determinant. Let p_i denote the maximum hopping range of projector P_i . We have thus shown that $p_i \leq nR$ for each i . We note that all the conclusions so far are valid independent of the system dimensionality and regardless of LTI, even though we will focus on 2D systems below.

For systems of infinite size and finite n , all the projectors are thus SL. However, for finite-sized systems, the projectors may not be SL (in the sense defined above for finite-sized systems) if nR is comparable to the system size. We note that $q := \max\{R, p_1, \dots, p_n\}$ is such that $q \leq nR$. All the projectors are SL if $2q + 2 \leq L_{x/y}$. This is therefore guaranteed if $2nR + 2 \leq L_{x/y}$.

In the subsequent sections, the following observation will be useful. If we group the unit cells so that $q \times q$ primitive

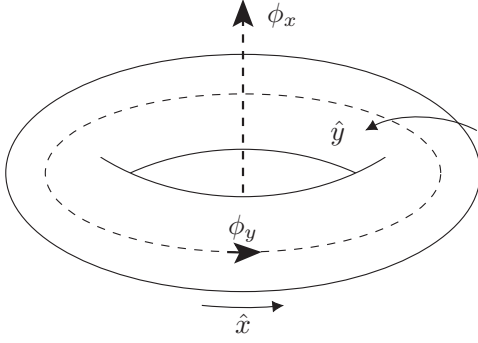


FIG. 1. Periodic boundary conditions are imposed on the system, so that it lies on a torus. Two magnetic fluxes ϕ_x and ϕ_y are then inserted through the two holes of the torus.

cells form a supercell, then the projectors and the Hamiltonian are NNN-hopping operators.

Spectrum is unchanged upon flux insertion. The spectrum of a Hamiltonian generally changes when magnetic fluxes are threaded through the two holes of a torus as shown in Fig. 1. However, for flat Hamiltonians that have SL projectors (and which further satisfy $L_{x,y} \geq 3q$ and $L_{x,y} \geq 2q + 2$), the spectrum is unchanged even after threading any amount of flux, as shown below.

Let us first consider the case when only ϕ_x is inserted, so that H changes to $H(\phi_x)$ following Peierls substitution [40]. We choose a gauge for the substitution that corresponds to twisted boundary conditions (TBCs) [6], so that only those hopping elements that cross the boundary at $x = L_x$ are modified. In other words, every hopping element that connects a cell with an x -coordinate in $\{L_x - R + 1, \dots, L_x\}$ to a cell with an x -coordinate in $\{1, 2, \dots, R\}$ is multiplied by a phase of $e^{-2\pi i \phi_x / \Phi_0}$, while hoppings in the reverse direction get multiplied by $e^{2\pi i \phi_x / \Phi_0}$. ($\Phi_0 = h/e$ is the magnetic flux quantum.) The rest of the hopping elements are unchanged in the substitution.

Similarly, we can implement TBCs for all the P_i as well if they are SL. Denoting the post-transformation operators by $P_i(\phi_x)$, we note that

$$H(\phi_x) = \sum_{i=1}^N E_i P_i(\phi_x). \quad (4)$$

While it is not immediately obvious, this equation is actually a spectral decomposition of $H(\phi_x)$ when $L_x \geq 3q$ [41].

This fact can also be used to obtain a spectral decomposition for the Hamiltonian $H(\phi_x, \phi_y)$, the Hamiltonian with both of the fluxes inserted, if $L_y \geq 3q$ and $L_y \geq 2q + 2$. The key idea is to treat $H(\phi_x)$ as the “initial Hamiltonian” and repeat the steps above. Specifically, we obtain $H(\phi_x, \phi_y)$ by applying TBCs to $H(\phi_x)$ along $\hat{y}$. Transforming the projectors similarly, so that $P_i(\phi_x) \rightarrow P_i(\phi_x, \phi_y)$, we get

$$H(\phi_x, \phi_y) = \sum_{i=1}^N E_i P_i(\phi_x, \phi_y). \quad (5)$$

Applying the proof from the Supplemental Material [41] again, we conclude that (5) is also a spectral decomposition. Thus, flat Hamiltonians have the interesting property of

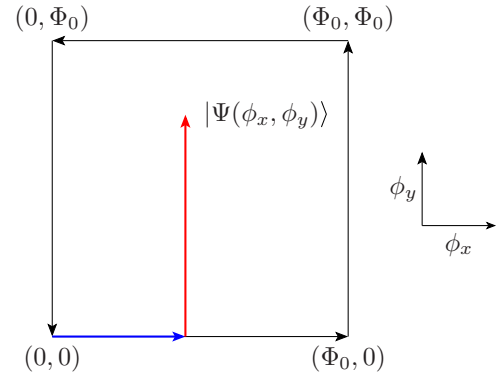


FIG. 2. The (ϕ_x, ϕ_y) parameter space with $\phi_x, \phi_y \in [0, \Phi_0]$. For any pair of values (ϕ_x, ϕ_y) , wave function $|\Psi(\phi_x, \phi_y)\rangle$ is obtained in two steps.

having an unchanging spectrum if $L_{x,y} \geq 2q + 2$ and $L_{x,y} \geq 3q$. Both these conditions are satisfied if $L_{x,y} \geq 3nR$ and $L_{x,y} \geq 2nR + 2$.

Many-body Chern number. We now turn to the computation of the Chern number. Let us recall that in the absence of interactions, the transverse conductivity of a system with a Hamiltonian that has LTI is proportional to the TKNN invariant [5], i.e., the Chern number of the occupied Bloch bands. More generally, the transverse conductivity is proportional to the Chern number of the (gapped) many-body ground state as a function of twist angles, or equivalently, the fluxes $\phi_{x,y}$ [6]. This many-body Chern number is expressed as

$$C = \frac{1}{2\pi} \int_0^{\Phi_0} \int_0^{\Phi_0} d\phi_x d\phi_y \nabla \times i \langle \Psi | \nabla_\phi \Psi \rangle, \quad (6)$$

where $|\Psi(\phi_x, \phi_y)\rangle$ is the many-body ground state of $H(\phi_x, \phi_y)$. Even for finite systems, the transverse conductivity has been shown to be proportional to this Chern number up to exponentially small errors in the system size [42,43]. We can straightforwardly generalize (6) in order to define a Chern number for an isolated flat band or energy in a flat Hamiltonian. Specifically, we use expression Eq. (6) but with $|\Psi\rangle$ taken to be the “flat band wave function,” i.e., the many-body wave function corresponding to only the flat band states being fully occupied, with the rest of the energies being unoccupied. We will henceforth use this definition.

Many-body wave functions as a function of the fluxes. Let us now compute the Chern number of a flat band with energy E and projector P . We assume that $L_{x,y} \geq 3nR$ so that the spectrum of a flat Hamiltonian is unaffected by the fluxes as shown above. Thus, for any ϕ_x, ϕ_y , while the state of the system corresponding to that energy level being fully filled is unique, the corresponding wave function $|\Psi(\phi_x, \phi_y)\rangle$ has a phase ambiguity. We will now provide a prescription that fixes the phase to obtain a global expression for $|\Psi(\phi_x, \phi_y)\rangle$ that can then be used to compute the Chern number (see fig. 2).

We will start with a brief outline of our approach. First, we obtain a particular type of orthogonal basis of single-particle wave functions spanning the energy E subspace when no fluxes are inserted. The Slater determinant then yields $|\Psi\rangle$ at zero flux. Next, we consider the case when only ϕ_x is inserted,

so that P transforms to $P(\phi_x)$. We show that wave functions obtained for $\phi_x = 0$, when modified appropriately, are eigenstates of $P(\phi_x)$ and thus span E at flux ϕ_x . The corresponding Slater determinant yields $|\Psi(\phi_x)\rangle$, which by construction is shown to be a smooth function of ϕ_x . In the last step, we insert flux ϕ_y in addition to ϕ_x . Specifically, for each value of ϕ_x , we follow a similar procedure to obtain a smooth family of wave functions as a function of ϕ_y . This finally results in a global expression for $|\Psi(\phi_x, \phi_y)\rangle$.

For the remainder of the manuscript, we will work within the size $q \times q$ supercell representation described previously. Thus, H and all P_i 's are then NNN operators in this representation, and we have a supercell grid of a size of at least 3×3 since we assumed $L_{x,y} \geq 3nR$. The wave functions obtained in this representation can always be expressed in the original primitive cell representation at the end.

The first step involves obtaining the flat band many-body wave function before inserting ϕ_x . First, we demote the y index to an orbital index so that P can be regarded as a NN projector on a 1D lattice with position coordinate x . Next, we will use the property that the image of a 1D SL projector can always be spanned by an orthogonal basis of compactly supported wave functions [26,27]. Specifically, we use the construction of generalized Wannier functions presented in Ref. [27] for 1D NN projectors. The wave functions so obtained are of two types: “monomers” and “dimers.” A monomer has support on only one cell, while a dimer has support only on two consecutive cells. (In the original 2D picture, each such wave function is strictly localized along $\hat{x}$ but is possibly delocalized along $\hat{y}$.)

Monomer-dimer bases have an important property that we call the “support property.” Specifically, whenever two wave functions from such a basis have nonzero support at a common cell, say x_0 , then their supports at x_0 are also mutually orthogonal.

We note further that monomer-dimer bases are not unique. For instance, multiplying any monomer or dimer in a given basis by a phase itself results in a different basis. Less trivial possibilities arise whenever any P_1^x hopping matrix [defined in (1)] has degenerate singular values; see Ref. [27]. However, for our purpose, it suffices to choose any valid monomer-dimer basis for P . For convenience, let us denote the chosen basis by a set $\mathcal{B}$. We obtain the flat band many-body wave function $|\Psi\rangle$ at zero flux by computing the Slater determinant of $\mathcal{B}$.

Next, we insert flux ϕ_x so that P now transforms to $P(\phi_x)$. The hopping matrices of P that connect $x = 0$ and 1 get transformed, with the rest being unaffected. The substitution can be summarized as

$$P_1^0 \xrightarrow{\phi_x} e^{-2\pi i \phi_x / \Phi_0} P_1^0$$

and $P_{-1}^1 \xrightarrow{\phi_x} e^{2\pi i \phi_x / \Phi_0} P_{-1}^1.$ (7)

While $\mathcal{B}$ is not an orthonormal basis for the image of $P(\phi_x)$, a simple modification of it does span the image. Consider two subsets of $\mathcal{B}$, namely $\mathcal{C}$, which consists only of those dimers that have support at both $x = 0$ and 1, and $\mathcal{U}$, which consists of all other wave functions from $\mathcal{B}$, so that $\mathcal{B} = \mathcal{U} \cup \mathcal{C}$. It is easy to see that every element of $\mathcal{U}$ is also an eigenvector of

$P(\phi_x)$ (with an eigenvalue of 1), since $P(\phi_x)$ is the same as P except for hopping elements between $x = 0$ and 1. However, the elements of $\mathcal{C}$ are not eigenvectors of $P(\phi_x)$. They can, however, be modified as follows. Consider a wave function $|w\rangle \in \mathcal{C}$. Let

$$|w\rangle = \alpha|x=0, \psi_0\rangle + \beta|x=1, \psi_1\rangle \quad (8)$$

for some support vectors $|\psi_{0,1}\rangle$. We now define $|w_{\phi_x}\rangle$ as follows:

$$|w_{\phi_x}\rangle := \alpha|x=0, \psi_0\rangle + e^{-i2\pi \phi_x / \Phi_0} \beta|x=1, \psi_1\rangle. \quad (9)$$

Using (7) and the fact that $P|w\rangle = |w\rangle$, we conclude that $P(\phi_x)|w_{\phi_x}\rangle = |w_{\phi_x}\rangle$. Let us denote by $\mathcal{C}_{\phi_x}$ the set obtained after modifying every element of $\mathcal{C}$ in this way, and define $\mathcal{B}_{\phi_x} := \mathcal{U} \cup \mathcal{C}_{\phi_x}$. Then, using the support property of monomer-dimer bases, we furthermore conclude that $\mathcal{B}_{\phi_x}$ is an orthogonal basis (and actually a monomer-dimer basis) for the image of $P(\phi_x)$. For any value of ϕ_x , the Slater determinant of $\mathcal{B}_{\phi_x}$ thus yields $|\Psi(\phi_x)\rangle$. Clearly, $|\Psi(\phi_x)\rangle$ is a smooth function of ϕ_x . Furthermore, since $\mathcal{B}_{\phi_x=\Phi_0}$ is the same as $\mathcal{B}_{\phi_x=0}$, we also find that $|\Psi(\phi_x=0)\rangle = |\Psi(\phi_x=\Phi_0)\rangle$.

Having obtained $|\Psi(\phi_x)\rangle$, we will now discuss the last step, i.e., obtaining the many-body wave function when ϕ_y is also inserted. To that end, for any value of ϕ_x , we first set $H(\phi_x)$ to be the “initial” flat Hamiltonian and then apply the entire procedure discussed above, but for the flux ϕ_y . [We thus obtain monomer-dimer states for each band projector $P_i(\phi_x, \phi_y)$, with the states being strictly localized along $\hat{y}$.] The Slater determinant of these wave functions, denoted $|\tilde{\Psi}_{\phi_x}(\phi_y)\rangle$, corresponds to the flat band at those flux values. Since the many-body state corresponding to the flat band is unique for any flux values, $|\Psi(\phi_x)\rangle$ and $|\tilde{\Psi}_{\phi_x}(\phi_y=0)\rangle$ must be the same up to a phase, so that $|\Psi(\phi_x)\rangle = e^{i\theta_{\phi_x}} |\tilde{\Psi}_{\phi_x}(\phi_y=0)\rangle$ for some $\theta_{\phi_x} \in \mathbb{R}$. Finally we define $|\Psi(\phi_x, \phi_y)\rangle := e^{i\theta_{\phi_x}} |\tilde{\Psi}_{\phi_x}(\phi_y)\rangle$.

We find that by construction, for all values of ϕ_x and ϕ_y ,

$$|\Psi(\phi_x, 0)\rangle = |\Psi(\phi_x, \Phi_0)\rangle,$$

and $|\Psi(0, \phi_y)\rangle = |\Psi(\Phi_0, \phi_y)\rangle.$ (10)

Using Stokes' theorem, we can write (6) as

$$C = \frac{1}{2\pi} \oint_{\Gamma} i \langle \Psi | \nabla_{\phi} \Psi \rangle \cdot d\phi, \quad (11)$$

where Γ is the boundary $(0, 0) \rightarrow (\Phi_0, 0) \rightarrow (\Phi_0, \Phi_0) \rightarrow (0, \Phi_0) \rightarrow (0, 0)$.

From (10), we conclude that $C = 0$. We note that the proof of the Chern triviality also extends to an energy that has a SL projector, regardless of the dispersiveness of the rest of the spectrum, a notion that is not always clear or applicable in the absence of LTI.

Conclusions. We showed that all bands of a two-dimensional strictly local (SL) flat Hamiltonian have a Chern number of zero. To that end, we showed that each band in such a system is described by a SL projector. We also showed that the spectrum of such Hamiltonians is unchanged even after flux threading in a toroidal geometry. We demonstrated all our results without the requirement of lattice-translational invariance (LTI), thereby going beyond existing no-go theorems concerning flat bands. Furthermore, we clarified the role of a

subtle condition that allows for a topological flat Hamiltonian in finite systems.

An important step of our proof involved showing that each band projector is strictly local. Since projection operators are equal-time Green's functions in noninteracting systems, generalizations to interacting cases might be possible. Our no-go theorem was proven for system sizes that are greater than a lower bound expressible in terms of the number of energies and maximum hopping range of the Hamiltonian. It would be interesting to obtain an improved lower bound. Due to the topology-localization correspondence, the existence of localized Wannier functions follows for flat Hamiltonians. Exploring no-go theorems in the presence of crystallographic symmetries, such as with fragile topology [44], would constitute an interesting direction for future work.

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